International Journal of Theoretical Physics, Vol. 44, No. 3, March 2005 (© 2005)
DOI: 10.1007/s10773-005-2988-7

Nonlinear Full Invariant of Compact Banach-Space Maps

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Received October 3, 2004; Accepted January 25, 2005

We characterize a nonlinear full invariant of compact Banach-space maps: Let $(X, \|.\|)$ and $(Y, \|.\|)$ be two Banach spaces and $P_C(X, Y)$ be all compact maps which map $(X, \|.\|)$ to $(Y, \|.\|)$. Then each weak operator-topology subseries-convergent series $\sum_i P_i$ in $P_c(X, Y)$ is also uniform-topology subseries-convergent iff each bounded map from $(X, \|.\|)$ to $(l^1, \|.\|_1)$ is a compact map. The necessary condition for each weak operator-topology subseries-convergent series $\sum_i P_i$ in $P_C(X, Y)$ to be also uniform-topology subseries-convergent is that $(X, \|.\|)$ and $(X^{'}, \|.\|)$ both contain no copy of c_0. This necessary condition is not sufficient.

KEY WORDS: Banach space; compact map; full invariant.
PACS: 02.10 By, 02.10 Gd

1. INTRODUCTION

A map $Q : (X, \|.\|) \to (Y, \|.\|)$ is said to be *a bounded (or compact, respectively) map* if for each bounded subset B of $(X, \|.\|)$, $Q(B)$ is a bounded (or compact, respectively) subset of $(Y, \|.\|)$.

Let $(X, \|.\|)$, $(Y, \|.\|)$ be two Banach spaces and $P_c(X, Y)$ the set of compact maps from $(X, \|.\|)$ to $(Y, \|.\|)$, $P_0(X, Y)$ the set of continuous compact polynomial operators from $(X, \|.\|)$ to $(Y, \|.\|)$, $K(X, Y)$ the set of continuous linear compact operator from $(X, \|.\|)$ to $(Y, \|.\|)$.

As is known, studying the invariants is a crucial topic in Mathematics and Physics. Li Ronglu, Cui Chengri, Cho Minhyung, Wu Junde and Lu Shijie proved several interesting linear full invariants (Li *et al.*, 1998; Wu and Li, 1999; Wu and Lu, 2002). In order to study nonlinear map-valued quantum measure theory, now, we characterize a nonlinear full invariant.

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Let *WOT*, *SOT* and *UOT* be the weak operator topology, strong operator topology and uniform operator topology on $P_C(X, Y)$, respectively, i.e. $\lim_\alpha P_\alpha = 0$ in the *WOT* $\Longleftrightarrow$ for each $x \in X$, $y' \in Y'$, $\lim_\alpha < T_\alpha x, y' >= 0$; $\lim_\alpha T_\alpha = 0$ in the *SOT* $\Longleftrightarrow$ for each $x \in X$, $\lim_\alpha T_\alpha(x) = 0$; $\lim_\alpha T_\alpha = 0$ in the *UOT* $\Longleftrightarrow$ for each bounded subset A of X, $\lim_\alpha T_\alpha x = 0$ uniformly with respect to $x \in A$.

It is clear that $WOT \subseteq SOT \subseteq UOT$.

Let τ_0 be a topology on $P_C(X, Y)$. A series $\sum_i P_i$ in $P_C(X, Y)$ is said to be τ_0-*subseries convergent* if for each sequence $\{k_j\}$ in $\mathbf{N}$, there exists an $P_0 \in P_C(X, Y)$ such that the series $\sum_j P_{k_j}$ is τ_0-converge to P_0.

If m_0 denotes the space of all scalar sequence (t_j) such that $\{t_j : j \in \mathbf{N}\}$ is a finite set. It is clear that $\sum_j P_j$ is τ_0-subseries convergent is equivalent to for each $(t_j) \in m_0$ there exists a $P_0 \in P_C(X, Y)$ such that the series $\sum_j t_j P_j$ is τ_0-convergent to P_0.

Definition 1. A property of $P_C(X, Y)$ is said to be *a full invariant* of $P_C(X, Y)$, if the property holds for some topology τ_0 of $P_C(X, Y)$ between *WOT* and *UOT*, then it also holds for all topologies τ of $P_C(X, Y)$ between *WOT* and *UOT*.

In order to prove our conclusion, we first need the following lemmas:

Lemma 1. (Wilansky, 1978) *$(l^1, \sigma(l^1, m_0))$, $(l^1, \sigma(l^1, l^\infty))$ and $(l^1, \|.\|_1)$ have the same bounded sets.*

Lemma 2. (Wu and Li, 2000) *If (X, τ_1) is a barrelled locally convex space, then the following are equivalent:*

(1) $(X', \beta(X', X))$ contains no copy of $(l^\infty, \|.\|_\infty)$.
(2) $(X', \beta(X', X))$ contains no copy of $(c_0, \|.\|_\infty)$.
(3) Each continuous linear operator $T : (X, \tau_1) \to (l^1, \|.\|_1)$ is a compact operator.

2. MAIN THEOREM AND PROOF

Now, we prove the following main result:

Theorem 1. *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be two Banach spaces and $Y \neq 0$. Then the subseries convergent property is a full invariant of $P_C(X, Y)$ iff each bounded map $T : (X, \|.\|) \to (l^1, \|.\|_1)$ is a compact map.*

Proof: *Sufficiency*. Without loss generality, let the series $\sum_i P_i$ in $P_C(X, Y)$ be weak operator topology subseries convergent. It follows from (Kalton, 1980) that $\sum_j P_j$ must be strong operator topology subseries convergent. Now, we show that if each bounded map $T : (X, \|.\|) \to (l^1, \|.\|_1)$ is a compact map, then $\sum_j P_j$ is uniform topology subseries convergent.

If not, there exists a subsequence $\{k_j\}$ of $\mathbf{N}$, a bounded subset B of $(X, \|.\|)$ and $P_0 \in P_C(X, Y)$ such that for each $x \in B$, the series $\sum_j P_{k_j} x$ is norm convergent to $P_0 x$, but $\sum_j P_{k_j} x$ does not converge to $P_0 x$ uniformly with respect to $x \in B$. Thus, there is an $\varepsilon_0 > 0$ such that for each $p \in \mathbf{N}$, there are $m, n \in \mathbf{N}$, $m \geq n > p$ and $x \in B$ satisfying

$$\left\| \sum_{j=n}^{m} P_{k_j} x \right\| \geq \varepsilon_0. \tag{1}$$

It follows from (1) inductively that we can obtain two sequences $n_1 \leq m_1 < n_2 \leq m_2 < \ldots < n_q \leq m_q < \ldots$ in $\mathbf{N}$ and $x_q \in B$ such that

$$\left\| \sum_{j=n_q}^{m_q} P_{k_j} x_q \right\| \geq \varepsilon_0, q \in \mathbf{N}.$$

By the Hahn–Banach theorem, there is a sequence $\{y_q'\}$ of Y' such that for each $q \in \mathbf{N}$, $\|y_q'\| \leq 1$ and

$$y_q'\left(\sum_{j=n_q}^{m_q} P_{k_j} x_q \right) \geq \varepsilon_0. \tag{2}$$

Let Y_0 be the linear closed hull of $\{P_j x_n : j, n \in \mathbf{N}\}$ in $(Y, \|.\|)$. Then $(Y_0, \|.\|)$ is a separable subspace of $(X, \|.\|)$. Thus, we can obtain a subsequence $\{y_{q_i}'\}$ of $\{y_q'\}$, without loss of generality, we may assume that $\{y_{q_i}'\}$ is just $\{y_q'\}$, and $y_0' \in Y'$ with $\|y_0'\| \leq 1$ such that for each $y \in Y_0$, $\lim_q y_q'(y) = y_0'(y)$ (Kothe, 1969).

For $P \in P_C(X, Y)$, we show that if $\{Px_n\} \subseteq Y_0$, then

$$\lim_q \sup_n \{|(y_q' - y_0')Px_n|\} = 0.$$

Otherwise, there exist a subsequence $\{y_{q_l}'\}$ of $\{y_q'\}$, a sequence $\{x_{k_l}\} \subseteq \{x_n\}$ and $\varepsilon_1 > 0$ such that

$$|(y_{q_l}' - y_0')Px_{k_l}| \geq \varepsilon_1, l \in \mathbf{N}. \tag{3}$$

Since $P \in P_C(X, Y)$, so the set $\{Px_{k_l}\}$ is relatively compact in $(Y, \|.\|)$. It follows from $\{Px_{k_l}\} \subseteq Y_0$ that $\{Px_{k_l}\}$ is a relatively compact subset of the norm space $(Y_0, \|.\|)$, and is also a relatively sequentially compact subset of $(Y_0, \|.\|)$. Thus, without loss of generality, we may assume that there exists a $y_0 \in Y_0$ such that $\{\|Px_{k_l} - y_0\|\}$ converges to 0. Note that

$$\begin{aligned} |(y_{q_l}' - y_0')Px_{k_l}| &\leq |(y_{q_l}' - y_0')(Px_{k_l} - y_0)| + |(y_{q_l}' - y_0')y_0| \\ &\leq \|y_{q_l}' - y_0'\| \|Px_{k_l} - y_0\| + |(y_{q_l}' - y_0')y_0|. \end{aligned}$$

It follows from $\|y'_{q_l} - y'_0\| \le 2$, $\{\|Px_{k_l} - y_0\|\} \to 0$ and $\{y'_{q_l}(y_0)\} \to y'_0(y_0)$ that

$$\lim_l (y'_{q_l} - y'_0)Px_{k_l} = 0.$$

This contradicts to (3). So the conclusion holds.

Furthermore, since the series $\sum_j P_j$ is strong operator topology subseries convergent, for each $(t_j) \in m_0$, there exists a $P \in P_C(X, Y)$ such that $\sum_j t_j P_j$ is strong operator topology convergent to P. So for each $y' \in Y'$ and $x \in X$,

$$\sum_j t_j \langle P_j x, y' \rangle = \langle Px, y' \rangle.$$

It is easy to prove that $(\langle P_i x, y' \rangle)_{i=1}^{\infty} \in l^1$. It follows from $\sum_j t_j \langle P_j x, y' \rangle = \langle Px, y' \rangle$ that the map: $x \to (\langle P_i x, y' \rangle)_{i=1}^{\infty}$ is a bounded map: $(X, \|.\|) \to (l^1, \sigma(l^1, m_0))$ and hence from Lemma 1 that it is also a bounded map of $(X, \|.\|) \to (l^1, \|.\|_1)$. Thus, the condition in Theorem 1 shows that $\{(\langle P_i x, y' \rangle)_{i=1}^{\infty} : x \in B\}$ is a relatively compact subset of $(l^1, \|.\|_1)$. So, it follows from the characteristic of the compact subsets of $(l^1, \|.\|_1)$ that the series $\sum_{j=1}^{\infty} t_j \langle P_j x, y' \rangle$ converges to $\langle Px, y' \rangle$ uniformly with respect to $x \in B$. Now, we consider the infinite matrix $[\sum_{i=n_j}^{m_j} y'_k P_i]_{kj}$. For each $j \in \mathbf{N}$, note that $\sum_{i=n_j}^{m_j} P_i \in P_C(X, Y)$ and $\{\sum_{i=n_j}^{m_j} P_i x_n\} \subseteq Y_0$, we have

$$\lim_k \sup_n \left| \sum_{i=n_j}^{m_j} (y'_k - y'_0) P_i(x_n) \right| = 0.$$

For each strictly increasing sequence of positive integers $\{j_r\}$, since the series $\sum_j P_j$ is strong operator topology subseries convergent, there exists $P_0 \in P_C(X, Y)$ such that the series $\sum_{r=1}^{\infty} \sum_{i=n_{j_r}}^{m_{j_r}} P_i$ is strong operator topology convergent to P_0. Therefore, the series $\sum_{r=1}^{\infty} \sum_{i=n_{j_r}}^{m_{j_r}} y'_k P_i(x)$ converges to $y'_k P_0(x)$ uniformly for $x \in B$. Thus we have

$$\sup_n \left\{ \left| \sum_{r=1}^{\infty} \sum_{i=n_{j_r}}^{m_{j_r}} y'_k P_i(x_n) - y'_k P_0(x_n) \right| \right\} = 0.$$

Note that $\{P_0 x_n\} \subseteq Y_0$ is obvious. Therefore, $\lim_k \sup_n \{|(y'_k - y'_0)P_0(x_n)|\} = 0$. It follows from Antosik–Mikusinski basic matrix theorem (Swartz, 1996) that

$$\lim_k \sup_n \left\{ \left| \sum_{i=n_k}^{m_k} y'_k P_i(x_n) \right| \right\} = 0.$$

This contradicts to (2) and the sufficiency is proved.

Necessity. Let P be a bounded map from $(X, \|.\|) \to (l^1, \|.\|_1)$. For $x \in X$, denote $Px = (P(x)_j)_{j=1}^{\infty}$. Pick $y \in Y, y \neq 0$ and define $P_j : X \to Y$ for

$P_j x = P(x)_j y$. It is obvious that $P_j \in P_C(X, Y)$. For each strictly increasing sequence $\{k_j\}$ in $\mathbf{N}$, let $P_0 x = \sum_j P(x)_{k_j} y$. Then $P_0 \in P_C(X, Y)$ and $\sum_j P_{k_j}$ is strong operator topology convergent to P_0. So $\sum_j P_{k_j}$ is uniform convergent to P_0. By the characteristic of compact sets in $(l^1, \|.\|_1)$ again that we can prove the map P is a compact map. The Theorem is proved. □

3. AN INTERESTING EXAMPLE

Let $(X, \|.\|)$ be a Banach space. A series $\sum_j x_j$ in $(X, \|.\|)$ is said to be a *weak unconditionally Cauchy series* if for each $f \in X'$, the series $\sum_j |f(x_j)| < \infty$. We may prove that $\sum_j x_j$ in $(X, \|.\|)$ is a weak unconditionally Cauchy series is equivalent to for each $(t_j) \in c_0$, the series $\sum_j t_j x_j$ is convergent in $(X, \|.\|)$ (Aizpuru and Perez-Fernandez, 2000), and if $\sum_j x_j$ in $(X, \|.\|)$ is a weak unconditionally Cauchy series, then for each bounded subset B of c_0, the set $\{\sum_j t_j x_j : (t_j) \in B\}$ is a bounded subset of X. If the series $\sum_j x_j$ in $(X, \|.\|)$ is norm topology subseries convergent, then $\sum_j x_j$ is said to be unconditionally convergent. M. Gonzalez and J.M. Gutierrez proved the following important conclusion (Gonzalez and Gutierrez, 2000):

Lemma 3. *Let P be a continuous polynomial operator of mappings $(X, \|.\|)$ into $(Y, \|.\|)$. Then the following assertions are equivalent:*

- (B) *Given a weak unconditionally Cauchy series $\sum_j x_j$ in $(X, \|.\|)$, if for each bounded subset B of c_0, the set $\{P(\sum_j t_j x_j) : (t_j) \in B\}$ is a relatively compact subset of $(Y, \|.\|)$, then the series $\sum_j x_j$ in $(X, \|.\|)$ is unconditionally convergent.*
- (D) *If the sequence $\{x_n\}$ in $(X, \|.\|)$ is equivalent to the c_0-basis, then there exists a bounded subset B of c_0 such that the set $\{P(\sum_j t_j x_j) : (t_j) \in B\}$ is not relatively compact in $(Y, \|.\|)$.*

It follows from Lemma 3 that if $(X, \|.\|)$ contains a copy of c_0, then there exists a continuous polynomial operator $P : (X, \|.\|) \to (Y, \|.\|)$ which is not a compact polynomial operator. Thus, it follows from Lemmas 2 and 3 that we have:

Theorem 2. *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be two Banach spaces. If each weak operator topology subseries convergent series $\sum_i T_i$ in $P_0(X, Y)$ is also uniform topology subseries convergent, then $(X, \|.\|)$ and $(X', \|.\|)$ both contain no copy of c_0.*

Since l^2 is a Hilbert space and $l^2 = (l^2)'$ contain both no copy of c_0, so the following example shows that the converse of Theorem 2 does not hold.

Example 1. Let $X = l^2$ and define the polynomial operator $P : l^2 \to l^1$ by $P(\{t_j\}) = \{t_j^2\}$. Then $P : l^2 \to l^1$ is a continuous polynomial operator which is not a compact polynomial operator.

Example 1 showed that the following problem is important and difficult:

Problem 1. Characterize the Banach space $(X, \|.\|)$ such that each continuous polynomial operator $P : (X, \|.\|) \to (l_1, \|.\|_1)$ is a compact polynomial operator.

ACKNOWLEDGMENT

This project was supported by Natural Science Fund of China (10471124) and Natural Science Fund of Zhejiang Province of China (M103057) and Natural Science Fund of China (10361005).

REFERENCES

Aizpuru, A. and Perez-Fernandez, J. (2000). *Acta Mathematica Hungary* **87**, 135–146.
Gonzalez, M. and Gutierrez, J. M. (2000). *Monatsh Math* **129**, 341–350.
Kalton, N. J. (1980). *Contemporary Mathematics* **2**, 91–100.
Kothe, G. (1969). *Topological Vector Spaces(I)*, Springer-Verlag New York.
Li, Ronglu, Cui, Chengri, and Cho, Minhyung (1998). *Chinese Annals of Mathematics* **19A**, 289–229.
Swartz, C. (1996). *Infinite Matrix and the Gliding Hump*, World Scientific Publication, Singapore.
Wilansky, A. (1978). *Modern Methods in Topological Vector Spaces*, McGraw-Hill, New York.
Wu, J. and Li, R. (1999). *Studia Science Mathematics Hungarary* **35**(3–4), 353–358.
Wu, Junde and Li, Ronglu (2000). *Taiwanese Journal of Mathematics* **4**, 253–259.
Wu, Junde and Lu, Shijie (2002). *Journal Mathematical Analysis Application* **270**(2), 397–404.